

SAMPLING THEOREMS FOR STURM LIOUVILLE PROBLEM WITH MOVING DISCONTINUITY POINTS

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ABSTRACT. In this paper, we investigate the sampling analysis for a new Sturm-Liouville problem with symmetrically located discontinuities which are defined to depending on a neighborhood of a midpoint of the interval. Also the problem has transmission conditions at these points of discontinuity and includes an eigenparameter in a boundary condition. We establish briefly the needed relations for the derivations of the sampling theorems and construct Green's function for the problem. Then we derive sampling representations for transforms whose kernels are either solutions or Green's functions.

1. INTRODUCTION

Throughout this study we consider Sturm Liouville problem:

$$(1.1) \quad \tau(u) := -u''(x) + q(x)u(x) = \lambda u(x), \quad x \in I,$$

with an eigenparameter dependent on a boundary condition;

$$(1.2) \quad B_a(u) := \beta_1 u(a) + \beta_2 u'(a) = 0,$$

$$(1.3) \quad B_b(u) := \lambda(\alpha'_1 u(b) - \alpha'_2 u'(b)) + (\alpha_1 u(b) - \alpha_2 u'(b)) = 0,$$

and transmission conditions at two points of discontinuity depending on a neighborhood of θ , that are $\theta_{-\varepsilon}$ and $\theta_{+\varepsilon}$;

$$(1.4) \quad T_{-\varepsilon}(u) := u(\theta_{-\varepsilon}-) - \delta u(\theta_{-\varepsilon}+) = 0,$$

$$(1.5) \quad T'_{-\varepsilon}(u) := u'(\theta_{-\varepsilon}-) - \delta u'(\theta_{-\varepsilon}+) = 0,$$

$$(1.6) \quad T_{+\varepsilon}(u) := \delta u(\theta_{+\varepsilon}-) - \gamma u(\theta_{+\varepsilon}+) = 0,$$

$$(1.7) \quad T'_{+\varepsilon}(u) := \delta u'(\theta_{+\varepsilon}-) - \gamma u'(\theta_{+\varepsilon}+) = 0,$$

where $I = [a, \theta_{-\varepsilon}) \cup (\theta_{-\varepsilon}, \theta_{+\varepsilon}) \cup (\theta_{+\varepsilon}, b]$; λ is a complex spectral parameter, $q(x)$ is a given real valued function which is continuous in $[a, \theta_{-\varepsilon})$, $(\theta_{-\varepsilon}, \theta_{+\varepsilon})$, and $(\theta_{+\varepsilon}, b]$ and has a finite limit $q(\theta_{-\varepsilon}\pm)$ and $q(\theta_{+\varepsilon}\pm)$; $\beta_i, \alpha_i, \alpha'_i, \delta, \gamma \in \mathbb{R}$ ($i = 1, 2$); $|\beta_1| + |\beta_2| \neq 0$, $\delta \neq 0$, $\gamma \neq 0$; $\theta := (a + b)/2$, $\theta_{\pm\varepsilon} \pm := (\theta \pm \varepsilon) \pm 0$, $0 < \varepsilon < (b - a)/2$ and

$$(1.8) \quad \rho := (\alpha'_1 \alpha_2 - \alpha_1 \alpha'_2) > 0.$$

In the literature, the Whittaker-Kotel'nikov-Shannon (WKS) sampling theorem and generalization of the WKS sampling theorem (see[1 – 3]) has been investigated

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extensively (see [4 – 8]). Sampling theorems associated with Sturm-Liouville problems were investigated in [9 – 13]. Also, [14 – 17] and [18 – 21] are the examples works in direction of sampling analysis associated with continuous and discontinuous eigenproblems, respectively. In [20] the author investigated the sampling analysis associated with discontinuous Sturm Liouville problems which has transmission conditions at the point of discontinuity and contains an eigenparameter in two boundary conditions. In the present study, we introduce a new Sturm Liouville problem which has symmetrically located discontinuities which are defined to depending on a neighborhood of a midpoint of the interval. ε is a parameter controlling the change of neighborhood process (it can be called tuning parameter) and by using the change of this ε parameter it's possible to determine points of discontinuity. That is, two points of discontinuity can be determined in interval $[a, b]$ for each ε value in interval $0 < \varepsilon < (b - a)/2$. This is the difference between the problem (1.1) – (1.7) and Sturm Liouville eigenvalue problem studied extensively in the literature (see [22 – 28]). The main result is that points of discontinuity can be determined and moved by changing ε parameter. A similar problem in more detail was presented in [28]. This study is the first to investigate that sampling analysis associated with eigenproblems with moving discontinuity points. To derive sampling theorems for the problem (1.1) – (1.7), we establish briefly the spectral properties and construct Green's function of the problem (1.1) – (1.7). Then we derive two sampling theorems using solutions and Green's function, respectively.

2. AN OPERATOR FORMULATION AND ASYMPTOTIC FORMULAS

To formulate a theoretic approach to the problem (1.1)-(1.7) we define the Hilbert space $H = L_2(a, b) \oplus \mathbb{C}$ with an inner product

$$(2.1) \quad \langle f(\cdot), g(\cdot) \rangle_H := \int_a^{\theta-\varepsilon} f(x) \overline{g(x)} dx + \delta^2 \int_{\theta-\varepsilon}^{\theta+\varepsilon} f(x) \overline{g(x)} dx + \gamma^2 \int_{\theta+\varepsilon}^b f(x) \overline{g(x)} dx + \frac{\gamma^2}{\rho} h \overline{k},$$

where $f(x) = \begin{pmatrix} f(x) \\ h \end{pmatrix}$, $g(x) = \begin{pmatrix} g(x) \\ k \end{pmatrix} \in H$, $f(\cdot), g(\cdot) \in L_2(a, b)$, $h, k \in \mathbb{C}$. For convenience we put

$$(2.2) \quad R(u) := \alpha_1 u(b) - \alpha_2 u'(b), \quad R'(u) := \alpha'_1 u(b) - \alpha'_2 u'(b).$$

Let $D(A) \subseteq H$ be the set of all $f(x) = \begin{pmatrix} f(x) \\ h \end{pmatrix} \in H$ such that f and f' are absolutely continuous on $[a, b]$ and $\tau(f) \in L_2(a, b)$, $h = R'(f)$, $B_a(f) = 0$, $T_{\pm\varepsilon}(f) = T'_{\pm\varepsilon}(f) = 0$. Define the operator $A : D(A) \rightarrow H$ by

$$(2.3) \quad A \begin{pmatrix} f(x) \\ R'(f) \end{pmatrix} = \begin{pmatrix} \tau(f) \\ -R(f) \end{pmatrix}, \quad \begin{pmatrix} f(x) \\ R'(f) \end{pmatrix} \in D(A).$$

The operator $A : D(A) \rightarrow H$ is equivalent to the eigenvalue problem (1.1)-(1.7) in the sense that the eigenvalues of A are exactly those of the problem (1.1)-(1.7).

We can prove in a manner similar to that of [23, 25, 26, 28] that A is symmetric in H , all eigenvalues of the problem are real.

Let $\phi_\lambda(\cdot)$ and $\chi_\lambda(\cdot)$ be two solutions of (1.1) as

$$(2.4) \quad \phi_\lambda(x) = \begin{cases} \phi_{-\varepsilon,\lambda}(x), & x \in [a, \theta_{-\varepsilon}), \\ \phi_{\varepsilon,\lambda}(x), & x \in (\theta_{-\varepsilon}, \theta_{+\varepsilon}), \\ \phi_{+\varepsilon,\lambda}(x), & x \in (\theta_{+\varepsilon}, b], \end{cases} \quad \chi_\lambda(x) = \begin{cases} \chi_{-\varepsilon,\lambda}(x), & x \in [a, \theta_{-\varepsilon}), \\ \chi_{\varepsilon,\lambda}(x), & x \in (\theta_{-\varepsilon}, \theta_{+\varepsilon}), \\ \chi_{+\varepsilon,\lambda}(x), & x \in (\theta_{+\varepsilon}, b], \end{cases}$$

satisfying the following conditions, respectively;

$$(2.5) \quad \phi_{-\varepsilon,\lambda}(a) = \beta_2, \quad \phi'_{-\varepsilon,\lambda}(a) = -\beta_1,$$

$$(2.6) \quad \phi_{\varepsilon,\lambda}(\theta_{-\varepsilon}) = \delta^{-1}\phi_{-\varepsilon,\lambda}(\theta_{-\varepsilon}-), \quad \phi'_{\varepsilon,\lambda}(\theta_{-\varepsilon}) = \delta^{-1}\phi'_{-\varepsilon,\lambda}(\theta_{-\varepsilon}-),$$

$$(2.7) \quad \phi_{+\varepsilon,\lambda}(\theta_{+\varepsilon}) = \delta\gamma^{-1}\phi_{\varepsilon,\lambda}(\theta_{+\varepsilon}-), \quad \phi'_{+\varepsilon,\lambda}(\theta_{+\varepsilon}) = \delta\gamma^{-1}\phi'_{\varepsilon,\lambda}(\theta_{+\varepsilon}-),$$

and

$$(2.8) \quad \chi_{+\varepsilon,\lambda}(b) = \lambda\alpha'_2 + \alpha_2, \quad \chi'_{+\varepsilon,\lambda}(b) = \lambda\alpha'_1 + \alpha_1,$$

$$(2.9) \quad \chi_{\varepsilon,\lambda}(\theta_{+\varepsilon}) = \gamma\delta^{-1}\chi_{+\varepsilon,\lambda}(\theta_{+\varepsilon}+), \quad \chi'_{\varepsilon,\lambda}(\theta_{+\varepsilon}) = \gamma\delta^{-1}\chi'_{+\varepsilon,\lambda}(\theta_{+\varepsilon}+),$$

$$(2.10) \quad \chi_{-\varepsilon,\lambda}(\theta_{-\varepsilon}) = \delta\chi_{\varepsilon,\lambda}(\theta_{-\varepsilon}+), \quad \chi'_{-\varepsilon,\lambda}(\theta_{-\varepsilon}) = \delta\chi'_{\varepsilon,\lambda}(\theta_{-\varepsilon}+).$$

These functions are entire in λ for all $x \in [a, b]$.

Let $W(\phi_\lambda, \chi_\lambda; x)$ be the Wronskian of ϕ_λ and χ_λ which is independent of x , since the coefficient of y' in the equation (1.1) is zero. Let

$$\begin{aligned} \omega(\lambda) &: = W(\phi_\lambda, \chi_\lambda; x) = \phi_\lambda(x)\chi'_\lambda(x) - \phi'_\lambda(x)\chi_\lambda(x) \\ &= \omega_{-\varepsilon}(\lambda) = \delta^2\omega_\varepsilon(\lambda) = \gamma^2\omega_{+\varepsilon}(\lambda). \end{aligned} \quad (2.11)$$

Then $\omega(\lambda)$ is an entire function of λ whose zeros are precisely the eigenvalues of the operator A . Using techniques similar of those established by Titchmarsh in [22], see also [25, 26, 28] the zeros of $\omega(\lambda)$ are real and simple and if $\lambda_n, n = 0, 1, 2, \dots$ denote the zeros of $\omega(\lambda)$, then the two component vectors

$$(2.12) \quad \Phi_n(x) := \begin{pmatrix} \phi_{\lambda_n}(x) \\ R'(\phi_{\lambda_n}) \end{pmatrix}$$

are the corresponding eigenvectors of the operator A satisfying the orthogonality relation

$$(2.13) \quad \langle \Phi_n(\cdot), \Phi_m(\cdot) \rangle_H = 0, \text{ for } n \neq m.$$

Here $\{\phi_{\lambda_n}(\cdot)\}_{n=0}^\infty$ will be the sequence of eigenfunctions of the problem (1.1)-(1.7) corresponding to the eigenvalues $\{\lambda_n\}_{n=0}^\infty$. We denote by

$$(2.14) \quad \Psi_n(x) := \frac{\Phi_n(x)}{\|\Phi_n(\cdot)\|_H} = \begin{pmatrix} \Psi_n(x) \\ R'(\Psi_n) \end{pmatrix}.$$

Let $k_n \neq 0$ be the real constants for which

$$(2.15) \quad \chi_{\lambda_n}(x) = k_n\phi_{\lambda_n}(x), \quad x \in I, \quad n = 0, 1, 2, \dots$$

The asymptotics of the eigenvalues and eigenfunctions can be derived similar to the classical techniques of [23, 25, 26, 28]. We state the results briefly.

$\phi_\lambda(\cdot)$ is the solution determined by equations (2.5)-(2.7) above then the following integral equations hold for $k = 0$ and $k = 1$:

$$\begin{aligned} \frac{d^k}{dx^k} \phi_{-\varepsilon, \lambda}(x) &= \beta_2 \frac{d^k}{dx^k} \left(\cos \sqrt{\lambda}(x-a) \right) - \frac{\beta_1}{\sqrt{\lambda}} \frac{d^k}{dx^k} (\sin \sqrt{\lambda}(x-a) + \\ &\quad \frac{1}{\sqrt{\lambda}} \int_a^x \frac{d^k}{dx^k} \left(\sin \sqrt{\lambda}(x-y) \right) q(y) \phi_{-\varepsilon, \lambda}(y) dy, \end{aligned} \quad (2.16)$$

$$\begin{aligned} \frac{d^k}{dx^k} \phi_{\varepsilon, \lambda}(x) &= \delta^{-1} \phi_{-\varepsilon, \lambda}(\theta_{-\varepsilon}-) \frac{d^k}{dx^k} \left(\cos \sqrt{\lambda}(x-\theta_{-\varepsilon}) \right) - \\ &\quad \frac{\delta^{-1}}{\sqrt{\lambda}} \phi'_{-\varepsilon, \lambda}(\theta_{-\varepsilon}-) \frac{d^k}{dx^k} (\sin \sqrt{\lambda}(x-\theta_{-\varepsilon}) + \\ &\quad \frac{1}{\sqrt{\lambda}} \int_{\theta_{-\varepsilon}}^x \frac{d^k}{dx^k} \left(\sin \sqrt{\lambda}(x-y) \right) q(y) \phi_{\varepsilon, \lambda}(y) dy, \end{aligned} \quad (2.17)$$

$$\begin{aligned} \frac{d^k}{dx^k} \phi_{+\varepsilon, \lambda}(x) &= \delta \gamma^{-1} \phi_{\varepsilon, \lambda}(\theta_{+\varepsilon}-) \frac{d^k}{dx^k} \left(\cos \sqrt{\lambda}(x-\theta_{+\varepsilon}) \right) - \\ &\quad \frac{\delta \gamma^{-1}}{\sqrt{\lambda}} \phi'_{\varepsilon, \lambda}(\theta_{+\varepsilon}-) \frac{d^k}{dx^k} (\sin \sqrt{\lambda}(x-\theta_{+\varepsilon}) + \\ &\quad \frac{1}{\sqrt{\lambda}} \int_{\theta_{+\varepsilon}}^x \frac{d^k}{dx^k} \left(\sin \sqrt{\lambda}(x-y) \right) q(y) \phi_{+\varepsilon, \lambda}(y) dy, \end{aligned} \quad (2.18)$$

sufficiently large λ and $\phi_\lambda(\cdot)$ have the following asymptotic representations for $|\lambda| \rightarrow \infty$, which hold uniformly for $x \in I$:

$$(2.19) \quad \frac{d^k}{dx^k} \phi_{-\varepsilon, \lambda}(x) = \beta_2 \frac{d^k}{dx^k} \left(\cos \sqrt{\lambda}(x-a) \right) + O \left(\left(\sqrt{\lambda} \right)^{k-1} e^{|t|(x-a)} \right),$$

$$(2.20) \quad \frac{d^k}{dx^k} \phi_{\varepsilon, \lambda}(x) = \beta_2 \delta^{-1} \frac{d^k}{dx^k} \left(\cos \sqrt{\lambda}(x-a) \right) + O \left(\left(\sqrt{\lambda} \right)^{k-1} e^{|t|(x-a)} \right),$$

$$(2.21) \quad \frac{d^k}{dx^k} \phi_{+\varepsilon, \lambda}(x) = \beta_2 \gamma^{-1} \frac{d^k}{dx^k} \left(\cos \sqrt{\lambda}(x-a) \right) + O \left(\left(\sqrt{\lambda} \right)^{k-1} e^{|t|(x-a)} \right),$$

if $\beta_2 \neq 0$,

$$(2.22) \quad \frac{d^k}{dx^k} \phi_{-\varepsilon, \lambda}(x) = -\frac{\beta_1}{\sqrt{\lambda}} \frac{d^k}{dx^k} \left(\sin \sqrt{\lambda}(x-a) \right) + O \left(\left(\sqrt{\lambda} \right)^{k-2} e^{|t|(x-a)} \right),$$

$$(2.23) \quad \frac{d^k}{dx^k} \phi_{\varepsilon, \lambda}(x) = -\frac{\beta_1 \delta^{-1}}{\sqrt{\lambda}} \frac{d^k}{dx^k} \left(\sin \sqrt{\lambda}(x-a) \right) + O \left(\left(\sqrt{\lambda} \right)^{k-2} e^{|t|(x-a)} \right),$$

$$(2.24) \quad \frac{d^k}{dx^k} \phi_{+\varepsilon, \lambda}(x) = -\frac{\beta_1 \gamma^{-1}}{\sqrt{\lambda}} \frac{d^k}{dx^k} \left(\sin \sqrt{\lambda}(x-a) \right) + O \left(\left(\sqrt{\lambda} \right)^{k-2} e^{|t|(x-a)} \right),$$

if $\beta_2 = 0$.

Then we obtain four distinct cases for the asymptotic behaviour of $\omega(\lambda)$ as $|\lambda| \rightarrow \infty$, namely;

$$(2.25) \quad \omega(\lambda) = \begin{cases} \lambda \sqrt{\lambda} \alpha'_1 \beta_2 \gamma \sin \sqrt{\lambda} (b-a) + O(\lambda e^{|\lambda|(b-a)}), & \text{if } \beta_2 \neq 0, \alpha'_1 \neq 0, \\ \lambda \alpha'_2 \beta_2 \gamma \cos \sqrt{\lambda} (b-a) + O(\sqrt{\lambda} e^{|\lambda|(b-a)}), & \text{if } \beta_2 \neq 0, \alpha'_1 = 0, \\ \lambda \alpha'_1 \beta_1 \gamma \cos \sqrt{\lambda} (b-a) + O(\sqrt{\lambda} e^{|\lambda|(b-a)}), & \text{if } \beta_2 = 0, \alpha'_1 \neq 0, \\ -\sqrt{\lambda} \alpha'_2 \beta_1 \gamma \sin \sqrt{\lambda} (b-a) + O(e^{|\lambda|(b-a)}), & \text{if } \beta_2 = 0, \alpha'_1 = 0. \end{cases}$$

Consequently if $\lambda_0 < \lambda_1 < \dots$, are the zeros of $\omega(\lambda)$, then we have for sufficiently large n the following asymptotic formulas

$$(2.26) \quad \sqrt{\lambda_n} = \begin{cases} \frac{(n-1)\pi}{b-a} + O(n^{-1}), & \text{if } \beta_2 \neq 0, \alpha'_1 \neq 0, \\ \frac{(n-1/2)\pi}{b-a} + O(n^{-1}), & \text{if } \beta_2 \neq 0, \alpha'_1 = 0, \\ \frac{(n-1/2)\pi}{b-a} + O(n^{-1}), & \text{if } \beta_2 = 0, \alpha'_1 \neq 0, \\ \frac{n\pi}{b-a} + O(n^{-1}), & \text{if } \beta_2 = 0, \alpha'_1 = 0. \end{cases}$$

3. GREEN FUNCTION

To study the completeness of the eigenvectors of A , and hence the completeness of the eigenfunctions of the problem (1.1)-(1.7), we construct the resolvent of A as well as Green's function of the problem (1.1)-(1.7). We assume without any loss of generality that $\lambda = 0$ is not an eigenvalue of A . Now let $\lambda \in \mathbb{C}$ not be an eigenvalue of A and consider the inhomogenous problem for $f(x) = \begin{pmatrix} f(x) \\ f_1 \end{pmatrix} \in H$,

$$u(x) = \begin{pmatrix} u(x) \\ R'(u) \end{pmatrix} \in D(A),$$

$$(3.1) \quad (\lambda I - A) u(x) = f(x), \quad x \in I,$$

and I is the identity operator. Since

$$(3.2) \quad (\lambda I - A) u(x) = \lambda \begin{pmatrix} u(x) \\ R'(u) \end{pmatrix} - \begin{pmatrix} \tau(u) \\ -R(u) \end{pmatrix} = \begin{pmatrix} f(x) \\ f_1 \end{pmatrix}$$

then we have

$$(3.3) \quad (\lambda I - \tau) u(x) = f(x), \quad x \in I,$$

$$(3.4) \quad \lambda R'(u) + R(u) = f_1.$$

Now we can represent the general solution of homogeneous differential equation (1.1), appropriate to equation (3.3) in the following form:

$$u(x, \lambda) = \begin{cases} c_1 \phi_{-\varepsilon, \lambda}(x) + c_2 \chi_{-\varepsilon, \lambda}(x), & x \in [a, \theta_{-\varepsilon}), \\ c_3 \phi_{\varepsilon, \lambda}(x) + c_4 \chi_{\varepsilon, \lambda}(x), & x \in (\theta_{-\varepsilon}, \theta_{+\varepsilon}), \\ c_5 \phi_{+\varepsilon, \lambda}(x) + c_6 \chi_{+\varepsilon, \lambda}(x), & x \in (\theta_{+\varepsilon}, b], \end{cases}$$

in which c_i ($i = \overline{1, 6}$) are arbitrary constants. By applying the method of variation of the constants, we shall search the general solution of the non-homogeneous linear differential equation (3.3) in the following form:

$$(3.5) \quad u(x, \lambda) = \begin{cases} c_1(x, \lambda) \phi_{-\varepsilon, \lambda}(x) + c_2(x, \lambda) \chi_{-\varepsilon, \lambda}(x), & x \in [a, \theta_{-\varepsilon}), \\ c_3(x, \lambda) \phi_{\varepsilon, \lambda}(x) + c_4(x, \lambda) \chi_{\varepsilon, \lambda}(x), & x \in (\theta_{-\varepsilon}, \theta_{+\varepsilon}), \\ c_5(x, \lambda) \phi_{+\varepsilon, \lambda}(x) + c_6(x, \lambda) \chi_{+\varepsilon, \lambda}(x), & x \in (\theta_{+\varepsilon}, b], \end{cases}$$

where the functions $c_i(x, \lambda)$ ($i = \overline{1, 6}$) satisfy the linear system of equation

$$(3.6) \quad \begin{cases} c'_1(x, \lambda) \phi_{-\varepsilon, \lambda}(x) + c'_2(x, \lambda) \chi_{-\varepsilon, \lambda}(x) = 0, \\ c'_1(x, \lambda) \phi'_{-\varepsilon, \lambda}(x) + c'_2(x, \lambda) \chi'_{-\varepsilon, \lambda}(x) = f(x), \end{cases} \quad \text{for } x \in [a, \theta_{-\varepsilon})$$

$$(3.7) \quad \begin{cases} c'_3(x, \lambda) \phi_{\varepsilon, \lambda}(x) + c'_4(x, \lambda) \chi_{\varepsilon, \lambda}(x) = 0, \\ c'_3(x, \lambda) \phi'_{\varepsilon, \lambda}(x) + c'_4(x, \lambda) \chi'_{\varepsilon, \lambda}(x) = f(x), \end{cases} \quad \text{for } x \in (\theta_{-\varepsilon}, \theta_{+\varepsilon})$$

$$(3.8) \quad \begin{cases} c'_5(x, \lambda) \phi_{+\varepsilon, \lambda}(x) + c'_6(x, \lambda) \chi_{+\varepsilon, \lambda}(x) = 0, \\ c'_5(x, \lambda) \phi'_{+\varepsilon, \lambda}(x) + c'_6(x, \lambda) \chi'_{+\varepsilon, \lambda}(x) = f(x), \end{cases} \quad \text{for } x \in (\theta_{+\varepsilon}, b].$$

Since λ is not an eigenvalue and $\omega_{-\varepsilon}(\lambda) \neq 0$, $\omega_{\varepsilon}(\lambda) \neq 0$, $\omega_{+\varepsilon}(\lambda) \neq 0$, each of the linear systems in (3.6)-(3.8) have a unique solution which leads

$$(3.9) \quad \begin{cases} c_1(x, \lambda) = \frac{1}{\omega_{-\varepsilon}(\lambda)} \int_a^{\theta_{-\varepsilon}} \chi_{-\varepsilon, \lambda}(y) f(y) dy + c_1(\lambda), \\ c_2(x, \lambda) = \frac{1}{\omega_{-\varepsilon}(\lambda)} \int_a^x \phi_{-\varepsilon, \lambda}(y) f(y) dy + c_2(\lambda), \end{cases} \quad \text{for } x \in [a, \theta_{-\varepsilon})$$

$$(3.10) \quad \begin{cases} c_3(x, \lambda) = \frac{1}{\omega_{\varepsilon}(\lambda)} \int_{\theta_{-\varepsilon}}^{\theta_{+\varepsilon}} \chi_{\varepsilon, \lambda}(y) f(y) dy + c_3(\lambda), \\ c_4(x, \lambda) = \frac{1}{\omega_{\varepsilon}(\lambda)} \int_{\theta_{-\varepsilon}}^x \phi_{\varepsilon, \lambda}(y) f(y) dy + c_4(\lambda), \end{cases} \quad \text{for } x \in (\theta_{-\varepsilon}, \theta_{+\varepsilon})$$

$$(3.11) \quad \begin{cases} c_5(x, \lambda) = \frac{1}{\omega_{+\varepsilon}(\lambda)} \int_{\theta_{+\varepsilon}}^b \chi_{+\varepsilon, \lambda}(y) f(y) dy + c_5(\lambda), \\ c_6(x, \lambda) = \frac{1}{\omega_{+\varepsilon}(\lambda)} \int_{\theta_{+\varepsilon}}^x \phi_{+\varepsilon, \lambda}(y) f(y) dy + c_6(\lambda), \end{cases} \quad \text{for } x \in (\theta_{+\varepsilon}, b]$$

where $c_i(\lambda)$ ($i = \overline{1, 6}$) are arbitrary constants. Substituting (3.9)-(3.11) into (3.5), we obtain the solution of the equation (3.3)

$$(3.12) \quad u(x, \lambda) = \begin{cases} \frac{\phi_{-\varepsilon, \lambda}(x)}{\omega_{-\varepsilon}(\lambda)} \int_x^{\theta_{-\varepsilon}} \chi_{-\varepsilon, \lambda}(y) f(y) dy + \frac{\chi_{-\varepsilon, \lambda}(x)}{\omega_{-\varepsilon}(\lambda)} \int_a^x \phi_{-\varepsilon, \lambda}(y) f(y) dy + \\ c_1(\lambda) \phi_{-\varepsilon, \lambda}(x) + c_2(\lambda) \chi_{-\varepsilon, \lambda}(x), & x \in [a, \theta_{-\varepsilon}), \\ \frac{\phi_{\varepsilon, \lambda}(x)}{\omega_{\varepsilon}(\lambda)} \int_x^{\theta_{+\varepsilon}} \chi_{\varepsilon, \lambda}(y) f(y) dy + \frac{\chi_{\varepsilon, \lambda}(x)}{\omega_{\varepsilon}(\lambda)} \int_{\theta_{-\varepsilon}}^x \phi_{\varepsilon, \lambda}(y) f(y) dy + \\ c_3(\lambda) \phi_{\varepsilon, \lambda}(x) + c_4(\lambda) \chi_{\varepsilon, \lambda}(x), & x \in (\theta_{-\varepsilon}, \theta_{+\varepsilon}), \\ \frac{\phi_{+\varepsilon, \lambda}(x)}{\omega_{+\varepsilon}(\lambda)} \int_x^b \chi_{+\varepsilon, \lambda}(y) f(y) dy + \frac{\chi_{+\varepsilon, \lambda}(x)}{\omega_{+\varepsilon}(\lambda)} \int_{\theta_{+\varepsilon}}^x \phi_{+\varepsilon, \lambda}(y) f(y) dy + \\ c_5(\lambda) \phi_{+\varepsilon, \lambda}(x) + c_6(\lambda) \chi_{+\varepsilon, \lambda}(x), & x \in (\theta_{+\varepsilon}, b]. \end{cases}$$

Then from the boundary conditions (3.4) and (1.2) and the transmission conditions (1.4)-(1.7) we get

$$(3.13) \quad \begin{aligned} c_1(\lambda) &= \frac{1}{\omega_{\varepsilon}(\lambda)} \int_{\theta_{-\varepsilon}}^{\theta_{+\varepsilon}} \chi_{\varepsilon, \lambda}(y) f(y) dy + \frac{1}{\omega_{+\varepsilon}(\lambda)} \int_{\theta_{+\varepsilon}}^b \chi_{+\varepsilon, \lambda}(y) f(y) dy + \frac{f_1}{\omega_{+\varepsilon}(\lambda)}, \\ c_2(\lambda) &= 0, \quad c_3(\lambda) = \frac{1}{\omega_{+\varepsilon}(\lambda)} \int_{\theta_{+\varepsilon}}^b \chi_{+\varepsilon, \lambda}(y) f(y) dy + \frac{f_1}{\omega_{+\varepsilon}(\lambda)}, \\ c_4(\lambda) &= \frac{1}{\omega_{-\varepsilon}(\lambda)} \int_a^{\theta_{-\varepsilon}} \phi_{-\varepsilon, \lambda}(y) f(y) dy, \quad c_5(\lambda) = \frac{f_1}{\omega_{+\varepsilon}(\lambda)}, \\ c_6(\lambda) &= \frac{1}{\omega_{-\varepsilon}(\lambda)} \int_a^{\theta_{-\varepsilon}} \phi_{-\varepsilon, \lambda}(y) f(y) dy + \frac{1}{\omega_{\varepsilon}(\lambda)} \int_{\theta_{-\varepsilon}}^{\theta_{+\varepsilon}} \phi_{\varepsilon, \lambda}(y) f(y) dy. \end{aligned}$$

Substituting (3.13) and (2.11) into (3.12), then (3.12) can be written as
(3.14)

$$u(x, \lambda) = \begin{cases} \frac{\phi_{-\varepsilon, \lambda}(x)}{\omega(\lambda)} \int_x^{\theta_{-\varepsilon}} \chi_{-\varepsilon, \lambda}(y) f(y) dy + \frac{\chi_{-\varepsilon, \lambda}(x)}{\omega(\lambda)} \int_a^x \phi_{-\varepsilon, \lambda}(y) f(y) dy + \\ \frac{\delta^2 \phi_{-\varepsilon, \lambda}(x)}{\omega(\lambda)} \int_{\theta_{-\varepsilon}}^{\theta_{+\varepsilon}} \chi_{\varepsilon, \lambda}(y) f(y) dy + \frac{\delta^2 \phi_{-\varepsilon, \lambda}(x)}{\omega(\lambda)} \int_{\theta_{+\varepsilon}}^b \chi_{+\varepsilon, \lambda}(y) f(y) dy + \\ \frac{\gamma^2 f_1}{\omega(\lambda)} \phi_{-\varepsilon, \lambda}(x), & x \in [a, \theta_{-\varepsilon}), \\ \\ \frac{\delta^2 \phi_{\varepsilon, \lambda}(x)}{\omega(\lambda)} \int_x^{\theta_{+\varepsilon}} \chi_{\varepsilon, \lambda}(y) f(y) dy + \frac{\delta^2 \chi_{\varepsilon, \lambda}(x)}{\omega(\lambda)} \int_{\theta_{-\varepsilon}}^x \phi_{\varepsilon, \lambda}(y) f(y) dy + \\ \frac{\gamma^2 \phi_{\varepsilon, \lambda}(x)}{\omega(\lambda)} \int_{\theta_{+\varepsilon}}^b \chi_{+\varepsilon, \lambda}(y) f(y) dy + \frac{\chi_{\varepsilon, \lambda}(x)}{\omega(\lambda)} \int_a^{\theta_{-\varepsilon}} \phi_{-\varepsilon, \lambda}(y) f(y) dy + \\ \frac{\gamma^2 f_1}{\omega(\lambda)} \phi_{\varepsilon, \lambda}(x), & x \in (\theta_{-\varepsilon}, \theta_{+\varepsilon}), \\ \\ \frac{\gamma^2 \phi_{+\varepsilon, \lambda}(x)}{\omega(\lambda)} \int_x^b \chi_{+\varepsilon, \lambda}(y) f(y) dy + \frac{\gamma^2 \chi_{+\varepsilon, \lambda}(x)}{\omega(\lambda)} \int_{\theta_{+\varepsilon}}^x \phi_{+\varepsilon, \lambda}(y) f(y) dy + \\ \frac{\chi_{+\varepsilon, \lambda}(x)}{\omega(\lambda)} \int_a^{\theta_{-\varepsilon}} \phi_{-\varepsilon, \lambda}(y) f(y) dy + \frac{\delta^2 \chi_{+\varepsilon, \lambda}(x)}{\omega(\lambda)} \int_{\theta_{-\varepsilon}}^{\theta_{+\varepsilon}} \phi_{\varepsilon, \lambda}(y) f(y) dy + \\ \frac{\gamma^2 f_1}{\omega(\lambda)} \phi_{+\varepsilon, \lambda}(x), & x \in (\theta_{+\varepsilon}, b]. \end{cases}$$

Hence, we have

$$\begin{aligned} u(x) &= (\lambda I - A)^{-1} f(x) \\ &= \begin{pmatrix} \int_a^{\theta_{-\varepsilon}} G(x, y, \lambda) f(y) dy + \delta^2 \int_{\theta_{-\varepsilon}}^{\theta_{+\varepsilon}} G(x, y, \lambda) f(y) dy + \\ \gamma^2 \int_{\theta_{+\varepsilon}}^b G(x, y, \lambda) f(y) dy + \frac{\gamma^2}{\omega(\lambda)} f_1 \phi_\lambda(x) \end{pmatrix} \quad (3.15) \\ &\quad R'(u) \end{aligned}$$

where

$$(3.16) \quad G(x, y, \lambda) = \frac{1}{\omega(\lambda)} \begin{cases} \chi_\lambda(x) \phi_{-\varepsilon, \lambda}(y), & a \leq y \leq x \leq \theta_{-\varepsilon}, \\ \chi_\lambda(y) \phi_{-\varepsilon, \lambda}(x), & a \leq x \leq y \leq \theta_{-\varepsilon}, \\ \chi_\lambda(x) \phi_\lambda(y), & \theta_{-\varepsilon} \leq y \leq x \leq \theta_{+\varepsilon}, \\ \chi_\lambda(y) \phi_\lambda(x), & \theta_{-\varepsilon} \leq x \leq y \leq \theta_{+\varepsilon}, \\ \chi_{+\varepsilon, \lambda}(x) \phi_\lambda(y), & \theta_{+\varepsilon} \leq y \leq x \leq b, \\ \chi_{+\varepsilon, \lambda}(y) \phi_\lambda(x), & \theta_{+\varepsilon} \leq x \leq y \leq b, \end{cases}$$

is Green's function of the problem (1.1)-(1.7).

4. THE SAMPLING THEOREM

In this section we derive two sampling theorems associated with the problem (1.1)-(1.7). For convenience we may assume that the eigenvectors of A are real valued.

Theorem 1. *Consider the problem (1.1)-(1.7), and let*

$$(4.1) \quad \phi_\lambda(x) = \begin{cases} \phi_{-\varepsilon,\lambda}(x), & x \in [a, \theta_{-\varepsilon}), \\ \phi_{\varepsilon,\lambda}(x), & x \in (\theta_{-\varepsilon}, \theta_{+\varepsilon}), \\ \phi_{+\varepsilon,\lambda}(x), & x \in (\theta_{+\varepsilon}, b], \end{cases}$$

be the solution defined above. Let $g(\cdot) \in L_2(a, b)$ and

$$(4.2) \quad F(\lambda) = \int_a^{\theta_{-\varepsilon}} g(x) \phi_{-\varepsilon,\lambda}(x) dx + \delta^2 \int_{\theta_{-\varepsilon}}^{\theta_{+\varepsilon}} g(x) \phi_{\varepsilon,\lambda}(x) dx + \gamma^2 \int_{\theta_{+\varepsilon}}^b g(x) \phi_{+\varepsilon,\lambda}(x) dx.$$

Then $F(\lambda)$ is an entire function of exponential type $(b-a)$ that can be reconstructed from its values at the points $\{\lambda_n\}_{n=0}^\infty$ via the sampling formula

$$(4.3) \quad F(\lambda) = \sum_{n=0}^\infty F(\lambda_n) \frac{\omega(\lambda)}{(\lambda - \lambda_n) \omega'(\lambda_n)}.$$

The series (4.2) converges absolutely on \mathbb{C} and uniformly on compact subset of \mathbb{C} . Here $\omega(\lambda)$ is the entire function defined in (2.11).

Proof. ... □

Remark 1. To see that expansion (4.3) is a Lagrange type interpolation, we may replace $\omega(\lambda)$ by the canonical product

$$(4.26) \quad \tilde{\omega}(\lambda) = \begin{cases} \prod_{n=0}^\infty \left(1 - \frac{\lambda}{\lambda_n}\right), & \text{if zero is not an eigenvalue,} \\ \lambda \prod_{n=1}^\infty \left(1 - \frac{\lambda}{\lambda_n}\right), & \text{if } \lambda_0 = 0 \text{ is an eigenvalue.} \end{cases}$$

From Hadamard's factorization theorem, see [31], $\omega(\lambda) = h(\lambda) \tilde{\omega}(\lambda)$, where $h(\lambda)$ is an entire function with no zeros. Thus,

$$(4.27) \quad \frac{\omega(\lambda)}{\omega'(\lambda_n)} = \frac{h(\lambda) \tilde{\omega}(\lambda)}{h(\lambda_n) \tilde{\omega}'(\lambda_n)}$$

and (4.2), (4.3) remain valid for the function $F(\lambda)/h(\lambda)$. Hence

$$(4.28) \quad F(\lambda) = \sum_{n=0}^\infty F(\lambda_n) \frac{h(\lambda) \tilde{\omega}(\lambda)}{(\lambda - \lambda_n) h(\lambda_n) \tilde{\omega}'(\lambda_n)}.$$

We may redefine (4.2) by taking kernel $\phi_\lambda(\cdot)/h(\lambda) = \tilde{\phi}_\lambda(\cdot)$ to get

$$(4.29) \quad \tilde{F}(\lambda) = \frac{F(\lambda)}{h(\lambda)} = \sum_{n=0}^\infty \tilde{F}(\lambda_n) \frac{\tilde{\omega}(\lambda)}{(\lambda - \lambda_n) \tilde{\omega}'(\lambda_n)}.$$

The next theorem is devoted to give interpolation sampling expansions associated with the problem (1.1)-(1.7) for integral transforms whose kernels defined in terms

of Green's function (see [20, 32]). As we see in (3.16), Green's function $G(x, y, \lambda)$ of the problem (1.1)-(1.7) has simple poles at $\{\lambda_n\}_{n=0}^{\infty}$.

Define the function $G(x, \lambda)$ to be $G(x, \lambda) := \omega(\lambda) G(x, y_0, \lambda)$, where $y_0 \in I$ is a fixed point and $\omega(\lambda)$ is the function defined in equation (2.11) or it is the canonical product (4.26).

Theorem 2. *Let $g(\cdot) \in L_2(a, b)$ and $F(\lambda)$ the integral transform*

$$(4.30) \quad F(\lambda) = \int_a^{\theta-\varepsilon} G(x, \lambda) \bar{g}(x) dx + \delta^2 \int_{\theta-\varepsilon}^{\theta+\varepsilon} G(x, \lambda) \bar{g}(x) dx + \gamma^2 \int_{\theta+\varepsilon}^b G(x, \lambda) \bar{g}(x) dx.$$

Then $F(\lambda)$ is an entire function of exponential type $(b-a)$ which admits the sampling representation

$$(4.31) \quad F(\lambda) = \sum_{n=0}^{\infty} F(\lambda_n) \frac{\omega(\lambda)}{(\lambda - \lambda_n) \omega'(\lambda_n)}.$$

Series (4.31) converges absolutely on \mathbb{C} and uniformly on compact subset of \mathbb{C} .

Proof. ... □

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